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LETTER TO THE EDITOR

Exact calculation of the anomalous dimension of the diffusion coefficient for a model of a random walk in a random potential

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Abstract. Renormalisation of a model of classical diffusion in a random potential is analysed. It is shown that at two dimensions the one-loop expression for the anomalous dimension of the diffusion coefficient is perturbatively exact leading to the coupling-dependent value $1/\nu = 2 + g/4\pi$ of the exponent ν.

We consider the problem of a random walk in a random potential described by the equation

$$\dot{x}_m = -D_0 \frac{\partial \psi(x)}{\partial x_m} + \eta_m \qquad x_m = x_m(t) \qquad \eta_m = \eta_m(t)$$

where x is the position of the particle diffusing in the random potential ψ, and η is Gaussian noise with zero mean and the variance $\eta_m(t)\eta_n(t') = 2D_0\delta(t'-t)\delta_{mn}$, which defines the bare diffusion coefficient D_0. The correlations of the random potential are Gaussian with zero mean and the variance

$$\langle \psi(x)\psi(x')\rangle_0 = g_0(-\Delta)^{-1}(x-x') \tag{1}$$

where $\Delta = \nabla^2$ is the Laplace operator, and the (non-negative) bare coupling constant g_0 describes the strength of the disorder.

This model is a special case of the extensively studied model of a random walk in a random environment [1-3], and it has the remarkable property that the beta function corresponding to the renormalised coupling constant g is trivial [4] (i.e. all the loop contributions to it vanish). It has also been conjectured [3, 5] that for the anomalous dimension of the diffusion coefficient the one-loop result [2, 3] is exact. In this letter, we present a perturbative proof of this conjecture for the model presented above. Unfortunately, our argument does not apply to the generalisation of this model [5, 6], in which the random potential correlations are proportional to $(-\Delta)^{-(1+\alpha)}$ $(\alpha > 0)$, and it remains an open question whether or not this conjecture is valid for the generalised model.

For $t > 0$ and arbitrary initial conditions, the distribution function $P(x, t)$ of the position x of the random walker satisfies the Fokker-Planck equation

$$[\partial_t - D_0\partial_m(\partial_m\psi + \partial_m)]P \equiv LP = 0$$

and we shall construct a perturbation expansion for the retarded Green function L^{-1} of this equation. For convenience, we exclude the variable t by Fourier transformation:

$$G_\omega = L_\omega^{-1} \qquad L_\omega = -\mathrm{i}\omega - D_0\nabla(\nabla\psi + \nabla).$$

Averaging the functional integral representation of the Green function:

$$G_\omega(x, x') = \det L_\omega \int D\hat{\varphi} D\varphi\varphi(x)\tilde{\varphi}(x') \exp[-\tilde{\varphi}L_\omega\varphi]$$

over the Gaussian distribution (1) of the random potential ψ (the integral over x in the exponent here and in other similar formulae is implied), we arrive at a field theory with the action

$$S = -\frac{1}{2g_0}\nabla\psi\nabla\psi + \tilde{\varphi}[m_0 + \nabla(\nabla + \nabla\psi)]\varphi \qquad (2)$$

where we have scaled the fields φ and $\tilde{\varphi}$ so that $m_0 = \mathrm{i}\omega/D_0$, and omitted the term $\mathrm{Tr}\ln L_\omega$. The only effect of this term is to cancel graphs with closed loops of the $\varphi\tilde{\varphi}$ propagator, and we shall neglect the contribution of such graphs by convention. The averaged Green function $\langle G_\omega\rangle_0$ is simply related to the full $\varphi\tilde{\varphi}$ propagator G_0 of the field theory (2): $D_0\langle G_\omega\rangle_0 = G_0$.

It has been shown earlier [4] that the field theory (2) is multiplicatively renormalisable (this is not at all trivial), and that the beta function corresponding to the renormalised coupling constant g is trivial: $\beta(g) = -\varepsilon g$, where $\varepsilon = 2-3$ (d is the space dimensionality). In terms of the renormalised action, this is a consequence of the property that only one renormalisation constant Z_1 is needed for multiplicative renormalisation of the field theory (2), and the renormalised action is thus of the form

$$S_{\mathrm{R}} = -\frac{1}{2g\mu^{\varepsilon}}\nabla\psi\nabla\psi + \tilde{\varphi}[m + Z_1\nabla(\nabla + \nabla\psi)]\varphi \qquad (3)$$

where the scale setting parameter μ has been introduced. The anomolous dimension of the diffusion coefficient γ_{D} is related to the renormalisation constant Z_1 as follows

$$\gamma_{\mathrm{D}} = -\mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_1 \qquad (4)$$

where the subscript indicates that the partial derivative is taken with fixed values of the bare parameters. Using the connection between the full propagator G of the renormalised field theory (3) and the averaged Green function $\langle G_\omega\rangle_0$: $D\langle G_\omega\rangle_0 \equiv D_0 Z_1^{-1}\langle G_\omega\rangle_0 = G$, where D is the renormalised diffusion coefficient, and the corresponding renormalisation group equations [6] we obtain for the long-time behaviour of the mean-square displacement of the random walk $\langle x^2\rangle_0$ in two dimensions (up to non-perturbative effects)

$$\langle\overline{x^2(t)}\rangle_0 \propto t^{2/[2+\gamma_{\mathrm{D}}(g))} = t^{2\nu}$$

which relates the anomalous dimension γ_{D} to the exponent ν. Due to triviality of the beta function, the anomalous behaviour is not universal: the exponent ν depends on the renormalised coupling constant g, which is an arbitrary parameter for the same reason, and we choose $g = g_0$. We shall show that $\gamma_{\mathrm{D}}(g) = g/4\pi$ exactly in perturbation theory, which corresponds to subdiffusive behaviour for $g > 0$.

Since we shall make use of the two-dimensionality of the model in an essential way, we cannot use the dimensional regularisation scheme. Therefore, we regularise the field theory (2) by introduction of a cutoff parameter Λ at large momenta. The essential point is that the anomalous dimensions and the beta function do not depend on the renormalised mass m of the model [7]. Therefore, we may calculate them in the massless theory, and henceforth set $m = 0$. In this case, the normalisation conditions of Green functions are usually defined at some finite values of external momenta, which then determine the momentum scale μ of the renormalised theory. However, we shall be constructing the perturbation expansion for a slightly modified model, for which this procedure is not sufficient, and therefore introduce the scaling parameter μ as the infrared cutoff in the regularised $\psi\psi$ correlation function:

$$\langle\psi(\boldsymbol{x}_1)\psi(\boldsymbol{x}_2)\rangle_{\mathrm{reg}} = g\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^2}\frac{\exp(\mathrm{i}\boldsymbol{p}(\boldsymbol{x}_1-\boldsymbol{x}_2))}{\boldsymbol{p}^2}(\theta(\boldsymbol{p}^2-\mu^2)-\theta(\boldsymbol{p}^2-\Lambda^2)) \tag{5}$$

which also takes care of the ultraviolet regularisation of the field theory (2). The full propagator G of the massless renormalised field theory may obviously be found by averaging the solution $G_\psi(\boldsymbol{x},\boldsymbol{y})$ of the equation

$$Z_1\nabla[\nabla+\nabla\psi(\boldsymbol{x})]G_\psi(\boldsymbol{x},\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{6}$$

over the 'renormalised' distribution of the random potential ψ:

$$G(\boldsymbol{x}-\boldsymbol{y}) = \langle G_\psi(\boldsymbol{x},\boldsymbol{y})\rangle \equiv \int \mathrm{D}\psi\,\exp\left(\frac{1}{2g}\psi\nabla^2\psi\right)G_\psi(\boldsymbol{x},\boldsymbol{y}).$$

It is convenient to introduce the function R

$$R(\boldsymbol{x},\boldsymbol{y};\psi) \equiv Z_1\exp[\psi(\boldsymbol{x})]G_\psi(\boldsymbol{x},\boldsymbol{y})$$

for which from the equation (6) we obtain

$$\nabla\exp[-\psi(\boldsymbol{x})]\nabla R(\boldsymbol{x},\boldsymbol{y};\psi) = -\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{7}$$

Introducing a new field variable V:

$$V(\boldsymbol{x};\psi) \equiv \exp[-\psi(\boldsymbol{x})]-1 \tag{8}$$

we cast the differential equation (7) into the form

$$[\nabla^2+\nabla V(\boldsymbol{x};\psi)\nabla]R(\boldsymbol{x},\boldsymbol{y};\psi) = -\delta(\boldsymbol{x}-\boldsymbol{y})$$

the Fourier transformation of which yields (we use the same notation for Fourier transforms and originals V and R):

$$\boldsymbol{p}^2R(\boldsymbol{p},\boldsymbol{q};\psi)+\int\frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\boldsymbol{pk}V(\boldsymbol{p}-\boldsymbol{k};\psi)R(\boldsymbol{k},\boldsymbol{q};\psi) = (2\pi)^d\delta(\boldsymbol{p}+\boldsymbol{q}). \tag{9}$$

The averaged solution of this equation $\langle R\rangle$ is equal to the full $\varphi\tilde{\varphi}$ propagator of the auxiliary field theory with the action

$$\bar{S} = -\frac{1}{2g}\nabla\psi\nabla\psi-\nabla\tilde{\varphi}\exp(-\psi)\nabla\varphi. \tag{10}$$

Also in this field theory all closed loops of the $\varphi\tilde{\varphi}$ propagator are zero by definition. We regularise it using the regularised $\psi\psi$ correlation function (5). Denoting the functional average with the weight $\exp(\bar{S})$ by double angular brackets, we express the full propagator G of the original field theory (3) as

$$G(\boldsymbol{x}_1-\boldsymbol{x}_2) = Z_1^{-1}\langle\!\langle\exp[-\psi(\boldsymbol{x}_1)]\varphi(\boldsymbol{x}_1)\tilde{\varphi}(\boldsymbol{x}_2)\rangle\!\rangle. \tag{11}$$

Due to the close relation between field theories (3) and (10), it is natural to assume that their renormalisation properties are also connected. Nevertheless, this is not obvious *a priori*; therefore a separate analysis of the renormalisation of the field theory (10) is required. All the fields are dimensionless, but it turns out that only the one-particle irreducible (1PI) Green functions of the form $\Gamma_{\tilde{\varphi}\varphi\psi^n}$, $n = 0, 1, 2, \ldots$, are superficially divergent. This follows from the absence of closed $\varphi\tilde{\varphi}$ loops and the fact that due to the derivatives at the interaction vertices the real degrees of divergence of the 1PI graphs are reduced by unity for each external φ or $\tilde{\varphi}$ leg. Further, in spite of the presence of an infinite number of interaction vertices in the action (10), only one renormalisation constant is needed to make the theory finite in the limt $\Lambda \to \infty$. This is a consequence of the invariance of the action (10) with respect to the following transformation of fields:

$$\varphi \to \varphi \exp(a/2) \qquad \tilde{\varphi} \to \tilde{\varphi} \exp(a/2) \qquad \psi \to \psi + a$$

which leads to the Ward identities

$$\Gamma_{\tilde{\varphi}(x_1)\varphi(x_2)} = -\int \mathrm{d}y \Gamma_{\tilde{\varphi}(x_1)\varphi(x_2)\psi(y)}$$

and

$$\Gamma_{\tilde{\varphi}(x_1)\varphi(x_2)\psi(y_1)\ldots\psi(y_{n-1})} = -\int \mathrm{d}y_n \Gamma \tilde{\varphi}(x_1)\varphi(x_2)\psi(y_1)\ldots\psi(y_n) \qquad n = 2, 3, \ldots$$

from which it follows that the renormalisation constants of all the interaction vertices in the action (10) are equal to the renormalisation constant of the free-field term $\nabla\tilde{\varphi}\nabla\varphi$. This implies, in particular, that the beta function of the renormalised coupling constant g is trivial also for the model (10).

Let us introduce the generating functional $G(a, \tilde{a}, b)$ of the full Green functions of the field theory (10)

$$G(a, \tilde{a}, b) = \int \mathrm{D}\varphi \mathrm{D}\tilde{\varphi} \mathrm{D}\psi \exp(\bar{S} + \varphi a + \tilde{\varphi}\tilde{a} + \psi b) \tag{12}$$

where a, $\tilde{a}$ and b are, respectively, the source fields of φ, $\tilde{\varphi}$ and ψ, and the generating functional of the connected Green functions $W(a, \tilde{a}, b) = \ln G(a, \tilde{a}, b)$. For the expression on the right-hand side of the equation (11) we obtain in terms of these functionals

$$\begin{aligned} &\langle\!\langle \exp[-\psi(x_1)]\varphi(x_1)\tilde{\varphi}(x_2) \rangle\!\rangle \\ &\quad = \exp\left(-\frac{\delta}{\delta b(x_1)}\right) \frac{\delta^2 G(a, \tilde{a}, b)}{\delta a(x_1)\delta\tilde{a}(x_2)}\bigg|_{a=\tilde{a}=b=0} \\ &\quad = \exp\left(-\frac{\delta}{\delta b(x_1)}\right) \{ W_{a(x_1)\tilde{a}(x_2)}(0, 0, b) \exp(\tfrac{1}{2} b\langle\psi\psi\rangle b)\}|_{b=0} \end{aligned} \tag{13}$$

where we have denoted the derivatives of the functional W by corresponding subscripts, and used the identity $G(a, \tilde{a}, b)|_{a=\tilde{a}=0} = \exp(\frac{1}{2} b\langle\psi\psi\rangle b)$, which follows from the definition (12) of the generating functional.

The operator $\exp(-\delta/\delta b(x_1))$ shifts the functional argument b of the functional on the right-hand side of the equation (13): $b(x)\to b(x)-\delta(x-x_1)$, therefore

$$\langle\!\langle \exp[-\psi(x_1)]\varphi(x_1)\tilde{\varphi}(x_2)\rangle\!\rangle$$

$$=\exp(\tfrac{1}{2}\langle\psi(x_1)\psi(x_1)\rangle)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int \mathrm{d}y_1\ldots \mathrm{d}y_n\delta(y_1-x_1)\ldots\delta(y_n-x_1)$$

$$\times\frac{\delta^n W_{a(x_1)\tilde{a}(x_2)}(0,0,b)}{\delta b(y_1\ldots\delta b(y_n)}\bigg|_{b=0}. \tag{14}$$

Let us denote

$$W_{2,0}(x_1,x_2)\equiv W_2(x_1,x_2)\equiv W_{a(x_1)\tilde{a}(x_2)}(0,0,0)$$

$$W_{2,n}(x_1,x_2\ y_1,\ldots,y_n)\equiv\frac{\delta^n W_{a(x_1)\tilde{a}(x_2)}(0,0,b)}{\delta b(y_1)\ldots\delta b(y_n)}\bigg|_{b=0}.$$

Then from (11) and (14) we finally obtain

$$Z_1 G(x_1-x_2)=\exp(\tfrac{1}{2}\langle\psi(x_1)\psi(x_1)\rangle)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}W_{2,n}(x_1,x_2;y_1,\ldots,y_n)|_{y_i=x_1}. \tag{15}$$

We have already shown that all the Green functions $W_{2;n}$ with distinct arguments may be renormalised by a single renormalisation constant, and power counting shows that when the arguments corresponding to ψ fields coincide, no additional divergences appear. Further, we shall show below that even this single renormalisation constant of the auxiliary theory is in fact finite and, when this is taken into account, it follows from equation (15) that

$$G(p)=Z_1^{-1}\exp(\tfrac{1}{2}\langle\psi(x_1)\psi(x_1)\rangle_{\text{reg}})\frac{H(p\mu^{-1};g)}{p^2}$$

where we have denoted by H the Fourier transform of the renormalised function $-\nabla^2\Sigma_{n=0}^{\infty}[(-1)^n/n!]W_{2,n}|_{y_i=x_1}$, and introduced the regularised form (5) of the $\psi\psi$ correlator. Using the standard normalisation condition $[\partial G^{-1}(p^2)/\partial p^2]|_{p^2=\mu^2}=1$ we obtain the relation

$$Z_1=C\exp\left(\frac{g}{4\pi}\ln\frac{\Lambda}{\mu}\right)$$

where the constant C is independent of μ. From this relation and the definition (4) it follows that

$$\gamma_D=g/4\pi$$

which is a perturbatively exact result.

To complete the analysis, we have to prove that the auxiliary field theory (10) does not contain divergences in the limit $\Lambda\to\infty$. We have already shown that this field theory may be renormalised by a single renormalisation constant, which we shall extract from the simplest Green function $\langle\!\langle\varphi\tilde{\varphi}\rangle\!\rangle$. Let us introduce the matrix $\boldsymbol{T}$, which is the solution of the equation

$$T_{mn}(p,q;\psi)+\int\frac{\mathrm{d}k}{(2\pi)^d}V(p-k;\psi)P^{\parallel}_{ml}(k)T_{ln}(k,q;\psi)=\delta_{mn}V(p+q;\psi) \tag{16}$$

where $P^{\parallel}$ is the projection operator $P^{\parallel}_{ml}(\boldsymbol{k}) \equiv k_m k_l / k^2$. From the definition (8) of V it follows that

$$\frac{V(\boldsymbol{x};\,\psi)}{1+V(\boldsymbol{x};\,\psi)} = -V(\boldsymbol{x};\,-\psi)$$

and, using this relation, the convolution theorem and the identity $P^{\perp} = 1 - P^{\parallel}$, we obtain from (16) the equation

$$T_{mn}(\boldsymbol{p},\boldsymbol{q};\,\psi) + \int \frac{\mathrm{d}\Sigma}{(2\pi)^d}\, V(\boldsymbol{p}-\boldsymbol{k};\,-\psi) P^{\perp}_{ml}(\boldsymbol{k}) T_{ln}(\boldsymbol{k},\boldsymbol{q};\,\psi) = -\delta_{mn} V(\boldsymbol{p}+\boldsymbol{q};\,-\psi). \tag{17}$$

On the other hand, in two dimensions the projection operators are related through the transformation $\varepsilon P^{\parallel} \varepsilon^{-1} = P^{\perp}$, where ε_{mn} is the totally antisymmetric matrix with the normalisation $\varepsilon_{12} = 1$. Therefore, equation (17) may be cast into the form

$$[\varepsilon T(\boldsymbol{p},\boldsymbol{q};\,\psi)\varepsilon^{-1}]_{mn} + \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\, V(\boldsymbol{p}-\boldsymbol{k};\,-\psi) P^{\parallel}_{ml}(\boldsymbol{k}) [\varepsilon T(\boldsymbol{k},\boldsymbol{q};\,\psi)\varepsilon^{-1}]_{ln}$$
$$= -\delta_{mn} V(\boldsymbol{p}+\boldsymbol{q};\,-\psi),$$

which, together with equation (16), implies

$$\varepsilon T(\boldsymbol{p},\boldsymbol{q};\,\psi)\varepsilon^{-1} = -T(\boldsymbol{p},\boldsymbol{q};\,-\psi).$$

Averaging over ψ we obtain $\varepsilon\langle T\rangle\varepsilon^{-1} = -\langle T\rangle$, therefore

$$\mathrm{Tr}\langle T\rangle = 0. \tag{18}$$

The perturbative solution for $\langle T\rangle$ contains at most logarithmic primitive divergences, which must be of the form $C\delta_{mn}$, where C is a quantity, singular at the limit $\Lambda \to \infty$. From relation (18), however, it follows that this constant is zero, and the quantity $\langle T\rangle$ is finite after subtraction of possible subdivergences. Due to the connection between R and T

$$R(\boldsymbol{p},\boldsymbol{q};\,\psi) = \frac{(2\pi)^d \delta(\boldsymbol{p}+\boldsymbol{q})}{p^2} + \frac{p_m}{p^2} T_{mn}(\boldsymbol{p},\boldsymbol{q};\,\psi) \frac{q_n}{q^2}$$

which follows from equations (9) and (16), we conclude that this is true also for the quantity $\langle R\rangle = \langle\!\langle \varphi\tilde{\varphi}\rangle\!\rangle = W_2$, thus confirming that the field theory (10) is free of divergences.

In conclusion, we have shown that at the upper critical dimension $d = d_c$ the anomalous diffusion coefficient for the problem of random walk in a random potential is given perturbatively exactly by the one-loop contribution in the two-dimensional case $d_c = 2$. Therefore the exponent ν is equal to $\nu = (2 + g/4\pi)^{-1}$, which implies subdiffusive behaviour for $g > 0$. Two-loop calculations for the generalised model [6], in which the correlation function of the random potential is such that $d_c > 2$, suggest that this might be true also for the generalised model. Unfortunately, the present approach cannot be applied to this case, and new ideas are needed to clarify this question.

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